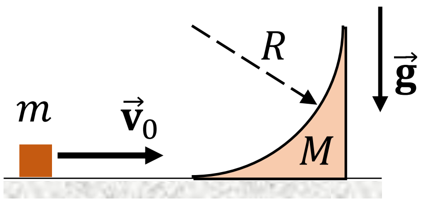
# PHYS 101 Final Exam Solution 2021-22-1

**1.** A block of mass $m$ and a wedge of mass $M$ in the shape of a quarter cylinder are free to move on a frictionless horizontal surface. The block moving with speed $v_0$ collides with the stationary wedge and smoothly starts to climb up its cylindrical frictionless surface.

(a) (7 Pts.) To what maximum height will the block reach?

(b) (7 Pts.) It is observed that the velocity of the block after it slides back to the horizontal surface is $-v_0/2$. Calculate the ratio $m/M$.

(c) (6 Pts.) What is the speed of the wedge after the block slides back down?




**Solution:**

(a) When the block reaches its maximum height $h_{\max}$ it will be momentarily at rest relative to the wedge, and hence, velocities of both the block and the wedge with respect to the horizontal surface will be the same $V_c$. Conservation of momentum implies

$$mv_0 = (m+M)V_c \quad \rightarrow \quad V_c = \frac{mv_0}{m+M}.$$

Since there is no friction between surfaces, total mechanical energy is also conserved. This means

$$\frac{1}{2}mv_0^2 = \frac{1}{2}(m+M)V_c^2 + mgh_{\max} \quad \rightarrow \quad \frac{1}{2}mv_0^2 = \frac{m^2v_0^2}{2(m+M)} + mgh_{\max} \quad \rightarrow \quad h_{\max} = \frac{v_0^2}{2g}\left(\frac{M}{m+M}\right).$$

(b) Since there is no energy loss, this is an elastic collision. Momentum conservation means

$$mv_0 = -\frac{1}{2}mv_0 + MV \quad \rightarrow \quad V = \frac{3mv_0}{2M},$$

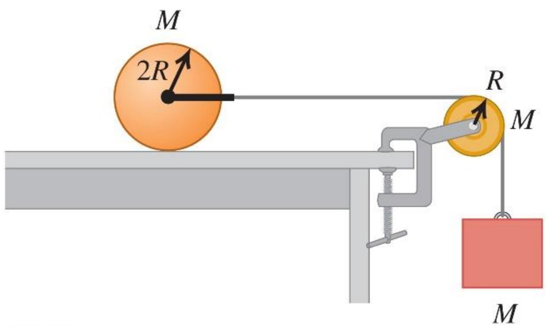
while energy conservation means

$$\frac{1}{2}mv_0^2 = \frac{1}{2}m\left(\frac{v_0}{2}\right)^2 + \frac{1}{2}MV^2 \quad \rightarrow \quad \frac{3}{4}mv_0^2 = \frac{9}{4}\frac{m^2v_0^2}{M} \quad \rightarrow \quad \frac{m}{M} = \frac{1}{3}.$$

(c)

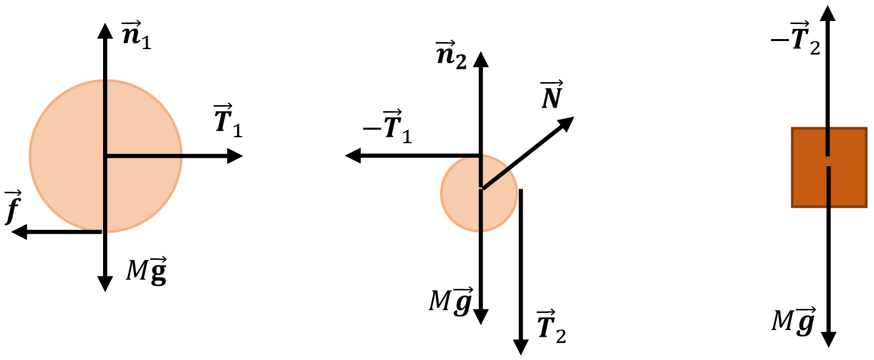
$$V = \frac{3mv_0}{2M} = \frac{v_0}{2}.$$

**2.** A uniform solid cylinder with mass $M$ and radius $2R$ rests on a horizontal table top. A string is attached by a yoke to a frictionless axle through the center of the cylinder so that the cylinder can rotate about the axle. The string runs over a disk-shaped pulley with mass $M$ and radius $R$ that is mounted on a frictionless axle through its center. A block of mass $M$ is suspended from the free end of the string. The string doesn't slip over the pulley surface, and the cylinder rolls without slipping on the table top. (For a cylinder of mass $M$ and radius $r$ moment of inertia about the symmetry axis is $I = (Mr^2)/2$.)




(a) (5 Pts.) Draw a free body diagram for all components of the system.
(b) (5 Pts.) Find the magnitude of the acceleration of the block after the system is released from rest.
(c) (5 Pts.) Find the tensions in the string.
(d) (5 Pts.) What will be the angular speed of the cylinder when the block falls a vertical distance $h$?

**Solution:** (a)




(b) (c) Let $a$ be the acceleration of the center of the rolling cylinder. Since the cylinder rolls without slipping, we have $a = 2R\alpha_1$, where $\alpha_1$ is the angular acceleration of the cylinder. Rolling withour slipping also means the downward acceleration of the block is also equal to $a$. If we let $\alpha_2$ denote the angular acceleration of the pulley, and write Newton's second law for each free body diagram, we have

$$T_1 - f = Ma\,, \qquad f(2R) = \frac{1}{2}M(2R)^2\left(\frac{a}{2R}\right), \qquad RT_2 - RT_1 = \frac{1}{2}MR^2\left(\frac{a}{R}\right), \qquad Mg - T_2 = Ma\,.$$

We have omitted non-contributing equations $\vec{n}_1 + M\vec{g} = 0$ (meaning that the cylinder's center has no vertical acceleration) and $\vec{n}_2 + \vec{N} + M\vec{g} = 0$ (meaning theat the center of the pulley is fixed). Solving equations

$$T_1 - f = Ma\,, \qquad f = \frac{1}{2}Ma\,, \qquad T_2 - T_1 = \frac{1}{2}Ma\,, \qquad Mg - T_2 = Ma\,,$$

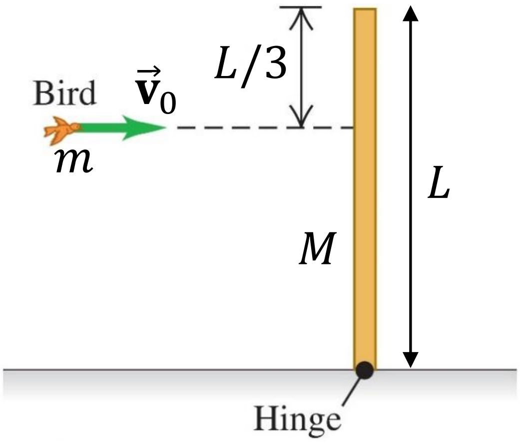
we find

$$a = \frac{1}{3}g\,, \qquad T_1 = \frac{1}{2}Mg\,, \qquad T_2 = \frac{2}{3}Mg\,.$$

(d) If we denote the time of fall by $t_f$, we have $h = at_f^2/2 \quad \rightarrow \quad t_f = \sqrt{2h/a}$. Therefore

$$\omega_f = \alpha_1 t_f = \frac{a}{2R}t_f \quad \rightarrow \quad \omega_f = \sqrt{\frac{hg}{6R^2}}\,.$$

**3.** A bird (mass $m$) is flying horizontally with speed $v_0$ suddenly flies into a stationary vertical bar of length $L$, hitting it at a distance $L/3$ below the top. The bar is uniform, has mass $M$, and is hinged (no friction) at its base. The collision stuns the bird so that it just drops vertically to the ground afterward. (Moment of inertia of a uniform bar about its center of mass is $I_{\mathrm{CM}} = ML^2/12$.)




(a) (4 Pts.) Which physical quantities are conserved in this collision?
(b) (8 Pts.) What is the angular velocity of the bar just after it is hit by the bird?
(c) (8 Pts.) What is the angular velocity of the bar just as it hits the ground?

**Solution:** (a) Angular momentum with respect to the hinge is conserved in the collison.

Moment of inertia of the bar about the hinge is found by using the parallel axis theorem.

$$I_H = I_{\mathrm{CM}} + M\left(\frac{L}{2}\right)^2 = \frac{1}{12}ML^2 + \frac{1}{4}ML^2 \quad \rightarrow \quad I_H = \frac{1}{3}ML^2 .$$

(b) Conservation of angular momentum with respect to the hinge means that we have

$$mv_0\left(L - \frac{L}{3}\right) = I_h\omega_{\mathrm{top}} \quad \rightarrow \quad \frac{2}{3}mv_0L = \frac{1}{3}ML^2\omega_{\mathrm{top}} \quad \rightarrow \quad \omega_{\mathrm{top}} = \frac{2mv_0}{ML} .$$

(c) After the collision, the total mechanical energy of the falling bar is conserved. This means

$$\frac{1}{2}I_h\omega_{top}^2 + Mg\left(\frac{L}{2}\right) = \frac{1}{2}I_h\omega_{bot}^2 \quad \rightarrow \quad \omega_{bot} = \sqrt{\omega_{top}^2 + \frac{3g}{L}} .$$

Using the result of part (a), we have

$$\omega_{bot} = \sqrt{\left(\frac{2mv_0}{ML}\right)^2 + \frac{3g}{L}} .$$

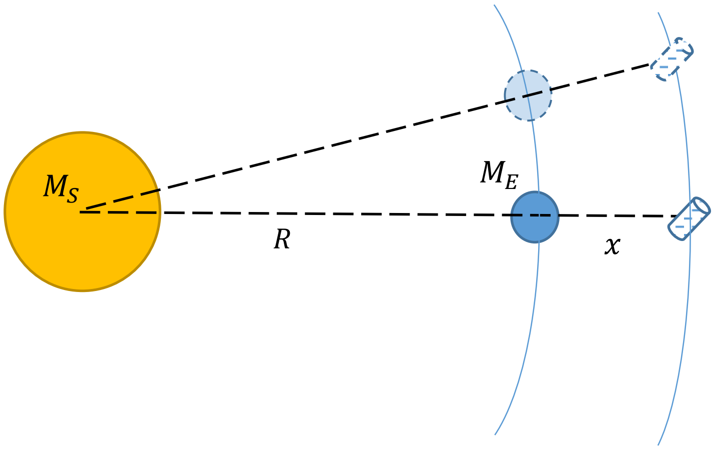
**4.** The next space telescope, called the James Webb Space Telescope (JWST) is now in space, trying to get to a special orbit. Because JSWT needs to be extremely cold it is put in an orbit around the Sun, in a way that it always remains in the shadow of the Earth. Assume that the Earth is making a circular trajectory around the Sun with radius $R$. JWST will also follow a circular trajectory around the Sun, with the same period of rotation as Earth, but it will be at a larger radius $R+x$ . The combined gravitational pull of the Sun and the Earth will keep JWST in this orbit. Mass of the Sun is $M_S$ , mass of the Earth is $M_E$, and their ratio is $M_E/M_S = 3\times 10^{-6}$. The mass of JWST is $m$, which is negligibly small compared to that of the Earth.

(a) (3 Pts.) Express the magnitude of the gravitational force on the JWST in terms of $M_S, M_E, m, R, x$ and Newton's constant $G$.

(b) (3 Pts.) Express the acceleration of JWST in terms of the same quantities.

(c) (10 Pts.) Find the numerical value of $\frac{x}{R}$. (Hint: you can assume that this quantity is very small and approximate $\left(1+\frac{x}{R}\right)^n \simeq 1+n\frac{x}{R}$ )

(d) (4 Pts.) What is the gravitational potential energy of JWST? (Express in terms of $M_S, M_E, m, R, x$ and $G$.)




**Solution:** (a) The combined gravitational force of the earth and the sun acting on JWST is

$$F = G\frac{M_S m}{(R+x)^2} + G\frac{M_E m}{x^2}.$$

(b) Angular speed $\omega$ of the earth in its orbit around the sun is found as

$$M_E R\omega^2 = G\frac{M_S M_E}{R^2} \quad \rightarrow \quad \omega^2 = G\frac{M_S}{R^3}.$$

If JWST is to remain in the shadow of the Earth, it must have the same angular speed. Therefore its centripetal accelertion must be

$$a = (R+x)\omega^2 = G\frac{M_S}{R^3}(R+x)\,.$$

(c) Using $F = ma$, we find

$$G\frac{M_S m}{(R+x)^2} + G\frac{M_E m}{x^2} = mG\frac{M_S}{R^3}(R+x) \quad \rightarrow \quad \left(1+\frac{x}{R}\right)^{-2} + \left(\frac{M_E}{M_S}\right)\frac{R^2}{x^2} = \left(1+\frac{x}{R}\right).$$

Using the approximation, we get

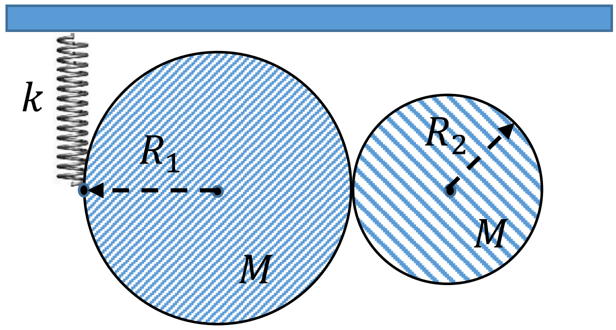
$$1 - 2\frac{x}{R} + \left(\frac{M_E}{M_S}\right)\frac{R^2}{x^2} = 1 + \frac{x}{R} \quad \rightarrow \quad \frac{x}{R} = \left(\frac{1}{3}\frac{M_E}{M_S}\right)^{1/3} \quad \rightarrow \quad \frac{x}{R} = 10^{-2} = 0.01\,.$$

(d)

$$U = -G\frac{M_S m}{R+x} - G\frac{M_E m}{x}.$$

**5.** Two cylinders of identical mass $M$ and radii $R_1$ and $R_2$ are pinned through their centers and can rotate around their centers without friction. The cylinders are in contact, and because there is friction between them, they can only roll without slipping with respect to each other. A spring with spring constant $k$ is affixed to the rim of the first cylinder as shown in the figure. The other end of the spring is fixed.

(Moment of inertia of a cylinder around its symmetry axis is $I_0 = \frac{1}{2}MR^2$.)




(a) (5 Pts.) If the first cylinder (one with radius $R_1$) has angular speed $\omega_1$ what would be the angular speed $\omega_2$ of the second cylinder?

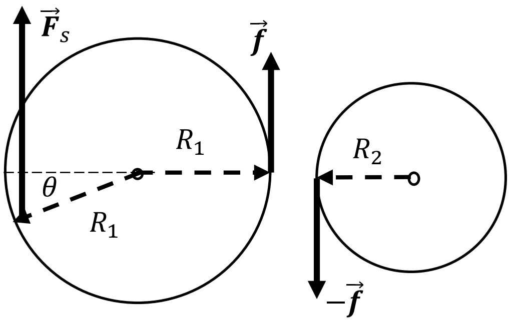
(b) (15 Pts.) Find the period of small oscillations of the system.

**Solution:**

(a) The cylinders roll without slipping on each other, and the point of contact is common to both cylinders. This means that the velocity of a point on the rim of one cylinder must be equal to the speed of a point on the rim on the rim of the other. That is

$$R_1\omega_1 = R_2\omega_2 \quad \rightarrow \quad \omega_2 = \frac{R_1}{R_2}\omega_1 \, .$$

The two angular velocities are opposite in direction.




(b) Free body diagrams for the cylinders:

The magnitude of the spring force is

$$F_s = kR_1\theta \, .$$

Writing the rotational form of Newton's second law for the first cylinder, we have

$$I_1\alpha_1 = R_1F_s - R_1f \quad \rightarrow \quad I_1\alpha_1 = R_1(kR_1\theta) - R_1f \, .$$

Writing the rotational form of Newton's second law for the second cylinder, we have

$$I_2\alpha_2 = -R_2f \quad \rightarrow \quad f = -\frac{I_2}{R_2}\alpha_2 \, .$$

Since we have $\alpha_2 = -R_1\alpha_1/R_2$, Newton's second law for the first cylinder is written as

$$I_1\alpha_1 = kR_1^2\theta - \frac{R_1^2}{R_2^2}I_2\alpha_1 \quad \rightarrow \quad \left(I_1 + \frac{R_1^2}{R_2^2}I_2\right)\alpha_1 = kR_1^2\theta \, .$$

Since the net torque on the first cylinder is in the decreasing $\theta$ direction, we have $\alpha_1 = -\frac{d^2\theta}{dt^2}$. So

$$\left(I_1 + \frac{R_1^2}{R_2^2}I_2\right)\frac{d^2\theta}{dt^2} + kR_1^2\theta = 0 \quad \rightarrow \quad \omega = \sqrt{\frac{kR_1^2}{I_1 + \frac{R_1^2}{R_2^2}I_2}} \, .$$

Using $I_1 = MR_1^2/2$ and $I_2 = MR_2^2/2$, we find

$$\omega = \sqrt{\frac{k}{M}} \, .$$